SOLUTION EXERCISE SHEET 1

Exercise 1. Assume that A is closed. To show that all limit points of convergent sequences $\{z_n\}_{n=0}^{\infty} \subset A$ lie in A, we assume for contradiction that there exists a sequence $\{\omega_n\}_{n=0}^{\infty} \subset A$ with $\omega_n \to \omega$ and $\omega \notin A$. Then, as the compliment A^c of A is open, there exists an $\varepsilon > 0$ such that the ball of radius ε centered around ω, denoted by $\mathbb{B}_\varepsilon(\omega)$, satisfies $\mathbb{B}_\varepsilon(\omega) \subset A^c$. This yields a contradiction.
For the other direction, we want to show that A^c is open. Assume for a contradiction that A^c is not open. This implies that there exists a $z \in A^c$ such that for every $n \in \mathbb{N}$ there exists a z_n in $A \cap \mathbb{B}_{\frac{1}{n}}(z)$. We get that $\{z_n\}_{n=0}^{\infty} \subset A$ and $z_n \to z$. Thus $z \in A \cap A^c$ which is again a contradiction.

Exercise 2. For $a \in U$ fixed consider the set

$$A := \{z \in U : \exists f \in C([0,1], U) : f(0) = a, f(1) = z\},$$

i.e. all point in U that can be reached from a via a continuous path. We claim that A is open, which follows immediately, since balls are path-connected sets. Likewise, it follows that $U \setminus A$ is open. Hence, given that $A \cap (U \setminus A) = \emptyset$ and $U = A \cap (U \setminus A)$ we see that necessarily A or $(U \setminus A)$ need to be empty. This yields the claim as $a \in A$.

Exercise 3. **(a)** One shows that the equality $T(z_1) = T(z_2)$ is equivalent to

$$T(z_1 - z_2) = 0.$$

Hence, by setting $z_1 - z_2 = x + iy$ as well as $\lambda = \zeta + i\omega$ and $\mu = \eta + i\nu$ one arrives at the linear system of equations

$$\begin{aligned} (\zeta + \eta)x + (\nu - \omega)y &= 0 \\ (\omega + \nu)x + (\zeta - \eta)y &= 0. \end{aligned}$$

In matrix form, this equation reads as

$$\begin{pmatrix} \zeta + \eta & \nu - \omega \\ \nu + \omega & \zeta - \eta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Furthermore, one readily computes

$$\det \begin{pmatrix} \zeta + \eta & \nu - \omega \\ \nu + \omega & \zeta - \eta \end{pmatrix} = \zeta^2 - \eta^2 - \nu^2 + \omega^2 = \lambda\overline{\lambda} - \mu\overline{\mu}$$

and the claim follows.
(b) Assume without loss of generality that $|z| = 1$. Then

$$\begin{aligned} |T(z)|^2 &= (\lambda z + \mu \overline{z})\overline{(\lambda z + \mu \overline{z})} \\ &= |\lambda|^2 + \lambda\overline{\mu} z^2 + \overline{\lambda}\mu \overline{z}^2 + |\mu|^2 \end{aligned}$$

Then, one readily infers that $\lambda\mu = 0$ and $|\lambda + \mu| = 1$ yield $|T(z)| = 1$. For the other direction, assume that λ and μ are both non-zero and note that

$$+\lambda\overline{\mu}z^2 + \overline{\lambda}\mu\overline{z}^2 = 2\operatorname{Re}(\lambda\overline{\mu}z^2).$$

We set $\omega^2 = 1 + i\frac{\operatorname{Re}\lambda\overline{\mu}}{\operatorname{Im}\lambda\overline{\mu}}$ and $\eta = \frac{\sqrt{\omega}}{|\sqrt{\omega}|}$ (there are 2 choices for this square root, and it does not matter which one is chosen). Then

$$2\operatorname{Re}(\lambda\overline{\mu}\eta^2) = 0$$

which implies

$$|\lambda|^2 + |\mu|^2 = 1.$$

From this one easily infers that

$$2\operatorname{Re}(\lambda\overline{\mu}z^2) = 0$$

for all $z \in \mathbb{C}$ with $|z| = 1$ which yields the claim.

Exercise 4. Note that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|a_{n,k}| = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}|a_{n,k}|. \tag{0.1}$$

One proves 0.1 as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|a_{n,k}| = \sup_{N\in\mathbb{N}}\sum_{n=0}^{N}\left(\sup_{K\in\mathbb{N}}\sum_{k=0}^{K}|a_{n,k}|\right) = \sup_{N\in\mathbb{N}}\sup_{K\in\mathbb{N}}\sum_{n=0}^{N}\sum_{k=0}^{K}|a_{n,k}| = \sup_{(N,K)\in\mathbb{N}^2}\sum_{k=0}^{K}\sum_{n=0}^{N}|a_{n,k}|.$$

Assume now that $a_{n,k} \in \mathbb{R}$ for all $n, k \in \mathbb{N}_0$ One defines the positive and negative parts of $a_{n,k}$ as

$$a^+_{n,k} = \max\{0, a_{n,k}\} \qquad a^-_{n,k} = \max\{0, -a_{n,k}\}.$$

Then, by the comparison principle one has that both $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^+_{n,k}$ and $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^-_{n,k}$ converge and they agree. Furthermore, as $a^+_{n,k}$ and $a^-_{n,k}$ are positive on also has that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a^+_{n,k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}a^+_{n,k}$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a^-_{n,k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}a^-_{n,k}$$

thanks to the previous considerations. Consequently, the claim follows as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{n,k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(a^+_{n,k} - a^-_{n,k}\right).$$

For complex valued sequences one applies the same reasoning to the real and imaginary parts separately.